Durée: 2h

1. (6pts) Consider the surface of revolution (S) parametrized by:

$$\begin{array}{rcl} X(u,v) & = & (f(v)\cos u,\ f(v)\sin u,\ g(v)) \\ a & \leq & v \leq b, \quad 0 \leq u \leq 2\pi, \quad f(v) > 0 \end{array}$$

given by the rotation of the curve

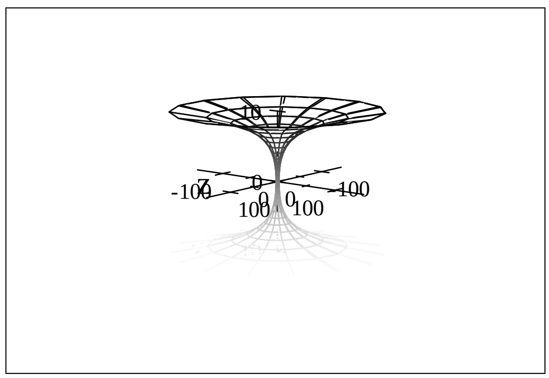
$$(C) : \left\{ \begin{array}{rcl} x & = & f(v) \\ z & = & g(v) \end{array} \right.$$

in the xoz plane about oz, where f and g are smooth and (C) is not necessirly a unit speed curve.

(a) Find the coefficients $E,\ F$ and G of the first fundamental form of X

(b) Consider in particular, the catenoid obtained by mean of the curve (C) :

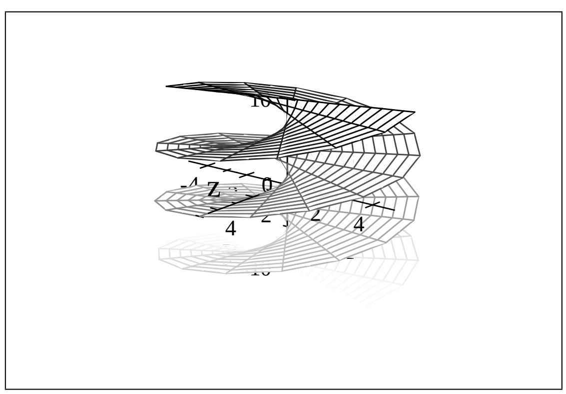
$$x = achv,\ z = av, \quad -\infty < v < \infty, \quad a \text{ is a constant}$$



Deduce the coefficients of its first fundamental form

(c) Let us consider now the helicoïd parametrized by

$$\bar{X}(\bar{u},\bar{v}) = (\bar{v}\cos\bar{u}, \bar{v}\sin\bar{u}, a\bar{u}), \qquad 0 < \bar{u} < 2\pi, \quad -\infty < \bar{v} < \infty$$



Recall that, a one to one map

$$\begin{array}{ccc}]0,2\pi[\times]-\infty,+\infty[& \xrightarrow{f} &]0,2\pi[\times]-\infty,+\infty[\\ (u,v) & \rightarrow & \bar{u} = u, \bar{v} = ashv \end{array}$$

is a change of parameters if the Jacobian $\dfrac{\partial(\bar{u},\bar{v})}{\partial(u,v)}$ is nonzero everywhere.

Verify that f defines a cahnge of parameters. Give the new parametrization of the helicoïd in terms of (u,v). deduce that the helicoid and .the catenoid are isometric

(d) Calculate the area of the domain of the catenoid corresponding to $0 < v < 1$.

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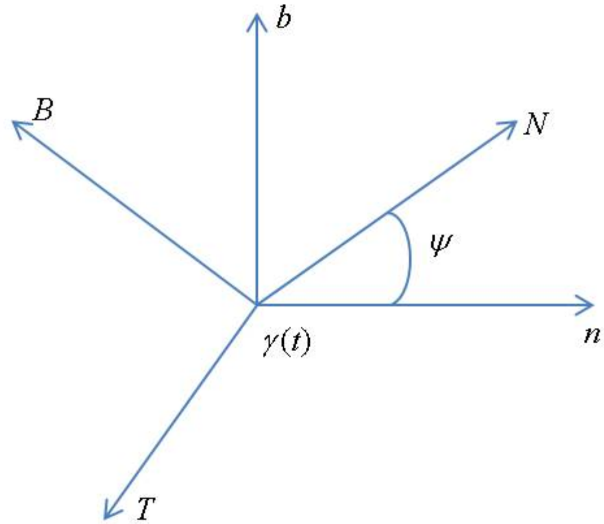
2. (6pts) Let γ be a unit speed curve on a surface patch $X(U)$ of a surface (S). Recall the following notations: N is the unit normal vector to (S) at the point $\gamma(t)$, T is the unit velocity vector of γ at $\gamma(t)$, n is the unit normal to γ at $\gamma(t)$, $b = T \wedge n$ is the binormal vector, $B = T \wedge N$, ψ is the angle between γ'' and N .



(a) Show that
$$N = (\cos\psi)n + (\sin\psi)b; \qquad B = (\cos\psi)b - (\sin\psi)n$$

(b) Deduce that:
$$T' = k_n N - k_g B;\; N' = -k_n T + \tau_g B;\;\; B' = k_g T - \tau_g N$$
where k_n is the normal curvature of γ, k_g is the geodesic curvature, $\tau_g = \tau + \psi'$ is the geodesic torsion of γ

(c) show that $\gamma'' = 0$ if and only if N is parallel to b

(d) The curve γ is called asymptotic if its norma curvature is everywhere zero. Deduce from c) that any straight line $\gamma = p + tq,\; q$ is a unit vector, on a surface (S) is an asymptotic curve

(e) Show also that a curve γ with positive curvature is asymptotic if and only if its binormal b is parallel to N at any point of γ

(f) Show that the asymptotic curves on the surface (S)
$$X(u,v) = (u\cos v, u\sin v, \ln u)$$
are given by: $\ln u = \pm(v + c)$, c is an arbitrary constant

(g) Show that an asymptotic curve with positive curvature has : $\tau_g = \tau$

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3. (2pts) Let $\gamma(t) = X(u(t), v(t))$ be a unit speed curve on a surface patch (X, U) of a surface (S). Shaw that the normal curvature of γ is given by
$$k_n = Lu'^2 + 2Mu'v' + Nv'^2$$
where $Ldu^2 + 2Mdudv + Ndv^2$ is the second fundamental form of X

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4. (6pts) Consider the paraboloïd (S) parametrized by

$$X(u, v) = (u \cos v, u \sin v, u^2)$$

(a) Find the first and the second fundamental forms of X

(b) Deduce the principal curvatures and the principal vectors of X, and give them a geometric interpretation

(c) Find the line of curvature on (S), i.e the curve on (S) whose tangent vector is principal at every point

(d) Does (S) admit umbilic points? if yes find them!